Math 4550
Homework 2
Solutions

1 (a)

$\mathbb{Z}_4 = \{\bar{0}, \bar{1}, \bar{2}, \bar{3}\}$

$\bar{0}$ has order 1
since its the
identity

$$\left.\begin{array}{l}\bar{1}\\ \bar{1}+\bar{1}=\bar{2}\\ \bar{1}+\bar{1}+\bar{1}=\bar{3}\\ \bar{1}+\bar{1}+\bar{1}+\bar{1}=\bar{4}=\bar{0}\end{array}\right\}$$

$\bar{1}$ has
order
4

$$\left.\begin{array}{l}\bar{2}\\ \bar{2}+\bar{2}=\bar{4}=\bar{0}\end{array}\right\}$$

$\bar{2}$ has
order 2

$$\left.\begin{array}{l}\bar{3}\\ \bar{3}+\bar{3}=\bar{6}=\bar{2}\\ \bar{3}+\bar{3}+\bar{3}=\bar{9}=\bar{1}\\ \bar{3}+\bar{3}+\bar{3}+\bar{3}=\overline{12}=\bar{0}\end{array}\right\}$$

$\bar{3}$ has
order 4

$\boxed{1(b)}$ $\mathbb{Z}_5 = \{\bar{0}, \bar{1}, \bar{2}, \bar{3}, \bar{4}\}$

$\bar{0}$ has order 1 since it's the identity element

$\bar{1}$

$\bar{1}+\bar{1} = \bar{2}$

$\bar{1}+\bar{1}+\bar{1} = \bar{3}$

$\bar{1}+\bar{1}+\bar{1}+\bar{1} = \bar{4}$

$\bar{1}+\bar{1}+\bar{1}+\bar{1}+\bar{1} = \bar{5} = \bar{0}$

$\bar{1}$ has order 5

$\bar{2}$

$\bar{2}+\bar{2} = \bar{4}$

$\bar{2}+\bar{2}+\bar{2} = \bar{6} = \bar{1}$

$\bar{2}+\bar{2}+\bar{2}+\bar{2} = \bar{8} = \bar{3}$

$\bar{2}+\bar{2}+\bar{2}+\bar{2}+\bar{2} = \overline{10} = \bar{0}$

$\bar{2}$ has order 5

$\bar{3}$

$\bar{3}+\bar{3} = \bar{6} = \bar{1}$

$\bar{3}+\bar{3}+\bar{3} = \bar{9} = \bar{4}$

$\bar{3}+\bar{3}+\bar{3}+\bar{3} = \overline{12} = \bar{2}$

$\bar{3}+\bar{3}+\bar{3}+\bar{3}+\bar{3} = \overline{15} = \bar{0}$

$\bar{3}$ has order 5

$\bar{4}$

$\bar{4}+\bar{4} = \bar{8} = \bar{3}$

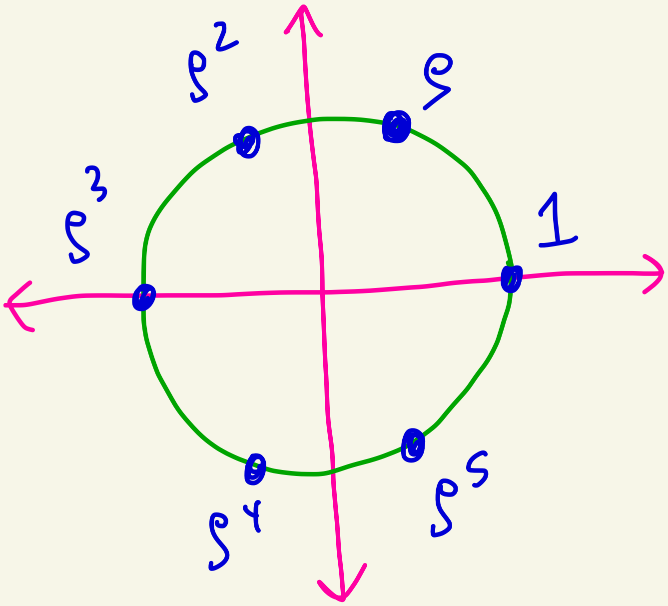
$\bar{4}+\bar{4}+\bar{4} = \overline{12} = \bar{2}$

$\bar{4}+\bar{4}+\bar{4}+\bar{4} = \overline{16} = \bar{1}$

$\bar{4}+\bar{4}+\bar{4}+\bar{4}+\bar{4} = \overline{20} = \bar{0}$

$\bar{4}$ has order 5

② $U_6 = \{1, \rho, \rho^2, \rho^3, \rho^4, \rho^5\}$



where $\rho = e^{\frac{2\pi i}{6}} = e^{\frac{\pi}{3} i}$.

and $\boxed{\rho^6 = 1}$

key formula for below

1 has order 1 since its the identity element

$\rho \neq 1$
$\rho^2 \neq 1$
$\rho^3 \neq 1$
$\rho^4 \neq 1$
$\rho^5 \neq 1$
$\rho^6 = 1$

ρ has order 6

$\rho^2 \neq 1$
$(\rho^2)^2 = \rho^4 \neq 1$
$(\rho^2)^3 = \rho^6 = 1$

So, ρ^2 has order 3.

$\rho^3 \neq 1$
$(\rho^3)^2 = \rho^6 = 1$

So, ρ^3 has order 2

$\rho^4 \neq 1$
$(\rho^4)^2 = \rho^8 = \rho^6 \cdot \rho^2 = \rho^2 \neq 1$
$(\rho^4)^3 = \rho^{12} = \rho^6 \cdot \rho^6 = 1 \cdot 1 = 1$

So, ρ^4 has order 3

$\rho^5 \neq 1$
$(\rho^5)^2 = \rho^{10} = \rho^6 \cdot \rho^4 = \rho^4 \neq 1$
$(\rho^5)^3 = \rho^{15} = \rho^6 \rho^6 \rho^3 = \rho^3 \neq 1$
$(\rho^5)^4 = \rho^{20} = \rho^6 \rho^6 \rho^6 \rho^2 = \rho^2 \neq 1$
$(\rho^5)^5 = \rho^{25} = (\rho^6)^4 \cdot \rho = 1 \cdot \rho = \rho \neq 1$
$(\rho^5)^6 = \rho^{30} = (\rho^6)^5 = 1^5 = 1$

So, ρ^5 has order 6

3 $D_6 = \{1, r, r^2, s, sr, sr^2\}$

Where $r^3 = 1$, $s^2 = 1$, $r^k s = s r^{-k} = s r^{3-k}$

1 has order 1 since its the identity

$r \neq 1$ $r^2 \neq 1$ $r^3 = 1$ So, r has order 3	$r^2 \neq 1$ $(r^2)^2 = r^4 = r^3 \cdot r = 1 \cdot r = r \neq 1$ $(r^2)^3 = r^6 = r^3 \cdot r^3 = 1 \cdot 1 = 1$ So, r^2 has order 3
$s \neq 1$ $s^2 = 1$ so, s has order 2	$sr \neq 1$ $(sr)^2 = srsr = ssr^{-1}r = s^2 \cdot 1 = 1 \cdot 1 = 1$ So, sr has order 2

$sr^2 \neq 1$

$(sr^2)^2 = sr^2sr^2 = ssr^{-2}r^2 = s^2 \cdot 1 = 1 \cdot 1 = 1$

So, sr^2 has order 2

$\boxed{4}$

$$\mathbb{Z}_8 = \{\bar{0}, \bar{1}, \bar{2}, \bar{3}, \bar{4}, \bar{5}, \bar{6}, \bar{7}\}$$

$$\langle \bar{2} \rangle = \{\bar{0}, \bar{2}, \underset{\bar{2}+\bar{2}}{\bar{4}}, \underset{\bar{2}+\bar{2}+\bar{2}}{\bar{6}}\}$$

stops at $\bar{6}$ since $\bar{2}+\bar{2}+\bar{2}+\bar{2} = \bar{8} = \bar{0}$

$$\langle \bar{4} \rangle = \{\bar{0}, \bar{4}\}$$

stops at $\bar{4}$ since $\bar{4}+\bar{4} = \bar{8} = \bar{0}$

$$\langle \bar{5} \rangle = \{\bar{0}, \bar{5}, \underset{\bar{5}+\bar{5}}{\bar{2}}, \underset{\bar{5}+\bar{5}+\bar{5}}{\bar{7}}, \underset{\bar{5}+\bar{5}+\bar{5}+\bar{5}}{\bar{4}}, \underset{\bar{5}+\bar{5}+\bar{5}+\bar{5}+\bar{5}}{\bar{1}}, \underset{\bar{5}+\bar{5}+\bar{5}+\bar{5}+\bar{5}+\bar{5}}{\bar{6}}, \underset{\bar{5}+\bar{5}+\bar{5}+\bar{5}+\bar{5}+\bar{5}+\bar{5}}{\bar{3}}\}$$

5

$$U_8 = \{1, \rho, \rho^2, \rho^3, \rho^4, \rho^5, \rho^6, \rho^7\}$$

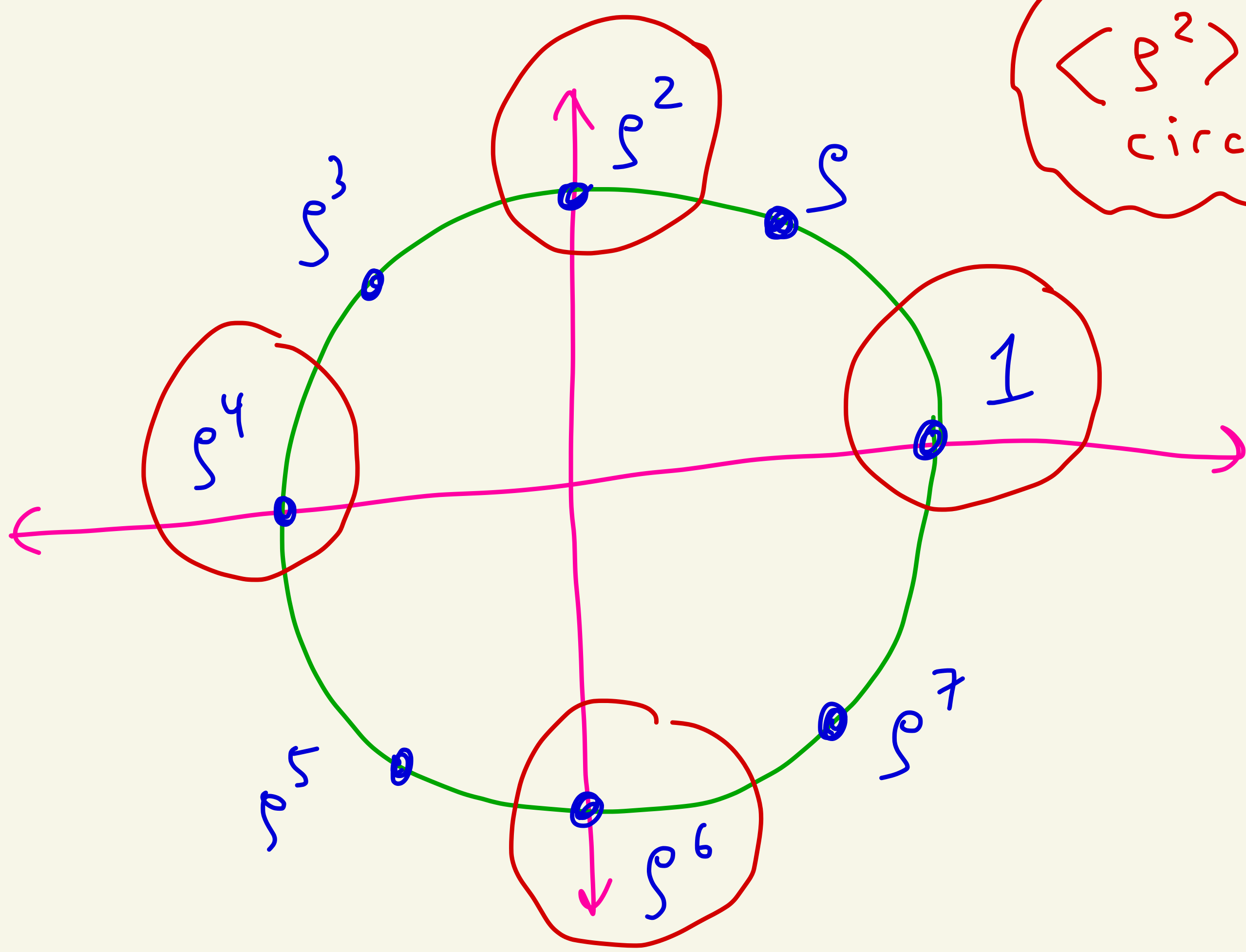
where $\rho = e^{\frac{2\pi i}{8}} = e^{\frac{\pi}{4} i}$ and $\rho^8 = 1$.

We want $\langle e^{2\pi i/4} \rangle = \langle \rho^2 \rangle$

We have

$$\langle \rho^2 \rangle = \{1, \rho^2, \underset{(\rho^2)^2}{\underset{\uparrow}{\rho^4}}, \underset{(\rho^2)^3}{\underset{\uparrow}{\rho^6}}\}$$

stops at ρ^6 since $(\rho^2)^4 = \rho^8 = 1$



$\langle \rho^2 \rangle$ is circled

$\boxed{⑥}$ $\mathbb{R}^* = \mathbb{R} - \{0\}$ is a group under multiplication.

$$\langle 3 \rangle = \{ 3^k \mid k \in \mathbb{Z} \}$$

$$= \{ \ldots, 3^{-4}, 3^{-3}, 3^{-2}, 3^{-1}, 1, 3, 3^2, 3^3, 3^4 \}$$

$$= \{ \ldots, \frac{1}{3^4}, \frac{1}{3^3}, \frac{1}{3^2}, \frac{1}{3}, 1, 3, 3^2, 3^3, 3^4, \ldots \}$$

7 (a)

$$\det(S) = \det\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = 0 \cdot 0 - (-1)(1) = 1 \neq 0$$

So, $S \in GL(2, \mathbb{R})$

---

7 (b)

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$$S^2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$S^3 = S \cdot S^2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$S^4 = S \cdot S^3 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$$

Thus, S has order 4 and

$$\langle S \rangle = \left\{ \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}_{\text{identity element } I}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\}$$

---

8 Note $\det(T) = \det\begin{pmatrix}1&1\\0&1\end{pmatrix} = 1-0 = 1$. So, $T \in GL(2,\mathbb{R})$.

$T = \begin{pmatrix}1&1\\0&1\end{pmatrix}$

$T^2 = \begin{pmatrix}1&1\\0&1\end{pmatrix}\begin{pmatrix}1&1\\0&1\end{pmatrix} = \begin{pmatrix}1&2\\0&1\end{pmatrix}$

$T^3 = T\cdot T^2 = \begin{pmatrix}1&1\\0&1\end{pmatrix}\begin{pmatrix}1&2\\0&1\end{pmatrix} = \begin{pmatrix}1&3\\0&1\end{pmatrix}$

$T^4 = T\cdot T^3 = \begin{pmatrix}1&1\\0&1\end{pmatrix}\begin{pmatrix}1&3\\0&1\end{pmatrix} = \begin{pmatrix}1&4\\0&1\end{pmatrix}$

$\vdots \quad \vdots$

The pattern is $T^n = \begin{pmatrix}1&n\\0&1\end{pmatrix}$ when $n \geq 1$.

$T^0 = I = \begin{pmatrix}1&0\\0&1\end{pmatrix}$. So, $T^0 = \begin{pmatrix}1&0\\0&1\end{pmatrix}$

$T^{-1} = \frac{1}{\det(T)}\begin{pmatrix}1&-1\\0&1\end{pmatrix} = \begin{pmatrix}1&-1\\0&1\end{pmatrix}$

$T^{-2} = T^{-1}T^{-1} = \begin{pmatrix}1&-1\\0&1\end{pmatrix}\begin{pmatrix}1&-1\\0&1\end{pmatrix} = \begin{pmatrix}1&-2\\0&1\end{pmatrix}$

$T^{-3} = T^{-1}T^{-2} = \begin{pmatrix}1&-1\\0&1\end{pmatrix}\begin{pmatrix}1&-2\\0&1\end{pmatrix} = \begin{pmatrix}1&-3\\0&1\end{pmatrix}$

$T^{-4} = T^{-1}T^{-3} = \begin{pmatrix}1&-1\\0&1\end{pmatrix}\begin{pmatrix}1&-3\\0&1\end{pmatrix} = \begin{pmatrix}1&-4\\0&1\end{pmatrix}$

$\vdots \quad \vdots$

The pattern is $T^n = \begin{pmatrix}1&n\\0&1\end{pmatrix}$ when $n < 0$.

Thus,

$\langle T\rangle = \left\{\ldots, \begin{pmatrix}1&-3\\0&1\end{pmatrix}, \begin{pmatrix}1&-2\\0&1\end{pmatrix}, \begin{pmatrix}1&-1\\0&1\end{pmatrix}, \begin{pmatrix}1&0\\0&1\end{pmatrix}, \begin{pmatrix}1&1\\0&1\end{pmatrix}, \begin{pmatrix}1&2\\0&1\end{pmatrix}, \begin{pmatrix}1&3\\0&1\end{pmatrix}, \ldots\right\}$

$= \left\{\begin{pmatrix}1&n\\0&1\end{pmatrix} \;\middle|\; n \in \mathbb{Z}\right\}$

⑨ $D_6 = \{1, r, r^2, s, sr, sr^2\}$

and $r^3 = 1$, $s^2 = 1$.

Since $r^3 = 1$ we know $r^{-1} = r^3 r^{-1} = r^2$

Let $H = \{1, s, sr, sr^2\}$ in D_6.

We use a table to show that H is not a subgroup of D_6.

H	1	s	sr	sr^2
1	1	s	sr	sr^2
s	s	1		
sr	sr	r^2	1	
sr^2	sr^2			1

We can stop filling in the table. We see that

$(sr)(s) = srs = ssr^{-1}$

$= 1r^{-1} = r^2 \notin H$

Since H is not closed under the operation, H is not a subgroup of D_6.

(10) $D_8 = \{1, r, r^2, r^3, s, sr, sr^2, sr^3\}$

and $r^4 = 1$, $s^2 = 1$, $r^k s = s r^{-k}$.

Let $H = \{1, r^2, s, sr^2\}$

We use a table to show that H is a subgroup of D_8.

H	1	r^2	s	sr^2
1	1	r^2	s	sr^2
r^2	r^2	1	sr^2	s
s	s	sr^2	1	r^2
sr^2	sr^2	s	r^2	1

Example calculations:

$$r^2 s = s r^{-2} = s r^4 r^{-2} = s r^2$$

$\boxed{r^4 = 1}$

$$r^2 (s r^2) = s r^{-2} r^2 = s$$

$$(s r^2) s = s r^2 s = s s r^{-2} = r^2$$

$$(s r^2)(s r^2) = s r^2 s r^2 = s s r^{-2} r^2$$

$$= s^2 \cdot 1 = 1$$

① $1 \in H$

② H is closed under the group operation by the table

③ H is closed under inversion by the table since $(r^2)^{-1} = r^2 \in H$, $s^{-1} = s \in H$, $(sr^2)^{-1} = sr^2 \in H$

By ①, ②, ③ we have that $H \trianglelefteq D_8$

11 Let $N = \left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \mid x \in \mathbb{R} \right\}$

proof that $N \trianglelefteq GL(2, \mathbb{R})$:

1 Setting $x = 0$ gives $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \in N$

2 Let $A = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$ be in N

where $a, b \in \mathbb{R}$.

Then,

$$AB = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & a+b \\ 0 & 1 \end{pmatrix}$$

which satisfies $a+b \in \mathbb{R}$.

So, $AB \in N$.

3 Let $C = \begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix} \in N$ where $c \in \mathbb{R}$.

Then, $C^{-1} = \begin{pmatrix} 1 & -c \\ 0 & 1 \end{pmatrix} \in N$ because $-c \in \mathbb{R}$.

By 1, 2, 3 we know $N \trianglelefteq GL(2, \mathbb{R})$

$\blacksquare$

12 Let $H = \{2x+3y \mid x, y \in \mathbb{Z}\}$

proof that $H \trianglelefteq \mathbb{Z}$:

1. Setting $x=0, y=0$ gives $0 = 2(0)+3(0) \in H$
2. Let $a = 2x_1 + 3y_1$ and $b = 2x_2 + 3y_2$ be in H where $x_1, y_1, x_2, y_2 \in \mathbb{Z}$.

   Then,

   $$a+b = 2x_1 + 3y_1 + 2x_2 + 3y_2$$
   $$= 2(x_1+x_2) + 3(y_1+y_2)$$

   is in H since $x_1+x_2, y_1+y_2 \in \mathbb{Z}$.
3. Let $c = 2x_3 + 3y_3$ be in H where $x_3, y_3 \in \mathbb{Z}$.

   Then, $-c = 2(-x_3) + 3(-y_3)$ is in H since $-x_3, -y_3 \in \mathbb{Z}$.

By 1, 2, 3 we have $H \trianglelefteq \mathbb{Z}$. ▨

13 $H = \{x \in G \mid x^2 = e\}$ and G is an abelian group.

Proof that $H \trianglelefteq G$:

① $e^2 = e$ gives that $e \in H$.

② Let $a, b \in H$.
Then $a^2 = e$ and $b^2 = e$.
Thus,

$$(ab)^2 = (ab)(ab) = abab$$
$$= aabb$$

since G is abelian

$$= a^2 b^2$$
$$= ee = e$$

Since $(ab)^2 = e$ we get that $ab \in H$

③ Let $c \in H$.
Then $c^2 = e$.
So, $c^{-2}c^2 = c^{-2}e$
Thus, $e = c^{-2}$.
So, $e = (c^{-1})^2$
Thus, $c^{-1} \in H$.

By ①, ②, ③ we have that $H \trianglelefteq G$. ▨

14

1. Since $H \trianglelefteq G$ we know $e \in H$.
   Since $K \trianglelefteq G$ we know $e \in K$.
   Thus, $e \in H \cap K$.

2. Let $a, b \in H \cap K$.
   So, $a \in H \cap K$ and $b \in H \cap K$.
   Thus, $a \in H, a \in K, b \in H, b \in K$.
   Since $H \trianglelefteq G$ and $a, b \in H$ we know $ab \in H$.
   Since $K \trianglelefteq G$ and $a, b \in K$ we know $ab \in K$.
   Thus, $ab \in H \cap K$.

3. Let $c \in H \cap K$.
   Then $c \in H$ and $c \in K$.
   Since $H \trianglelefteq G$ and $c \in H$ we know $c^{-1} \in H$.
   Since $K \trianglelefteq G$ and $c \in K$ we know $c^{-1} \in K$.
   Thus, $c^{-1} \in H \cap K$.

By 1, 2, 3 we know that $H \cap K \trianglelefteq G$.

$\boxed{//}$

(15) G is abelian, $H \trianglelefteq G$, $K \trianglelefteq G$
$HK = \{hk \mid h \in H, k \in K\}$

proof that $HK \trianglelefteq G$:

(1) Since $H \trianglelefteq G$ we know $e \in H$.
Since $K \trianglelefteq G$ we know $e \in K$.
Thus, $e = ee \in HK$.

(2) Let $a, b \in HK$.
Then $a = h_1 k_1$ and $b = h_2 k_2$
where $h_1, h_2 \in H$ and $k_1, k_2 \in K$.

We have
$$ab = h_1 k_1 h_2 k_2 = h_1 h_2 k_1 k_2$$
Since G is abelian

Since $h_1, h_2 \in H$ and $H \trianglelefteq G$ we know $h_1 h_2 \in H$.
Since $k_1, k_2 \in K$ and $K \trianglelefteq G$ we know $k_1 k_2 \in K$
Thus, $ab = (h_1 h_2)(k_1 k_2) \in HK$.

(3) Let $c \in HK$.
Then $c = hk$ where $h \in H$ and $k \in K$.
Since $h \in H$ and $H \trianglelefteq G$ we
know that $h^{-1} \in H$.

Since $k \in K$ and $K \trianglelefteq G$ we know that $k^{-1} \in K$.

formula from class

since G is abelian

Thus,

$$c^{-1} = (hk)^{-1} = k^{-1}h^{-1}$$

$$= h^{-1}k^{-1} \in HK$$

By ①, ②, ③ we know that $HK \trianglelefteq G$.

$\boxed{//}$

16

1. We know that $ey = y = ye$ for all $y \in G$. Thus, $e \in Z(G)$.

2. Let $a, b \in Z(G)$.
   Then $ay = ya$ for all $y \in G$
   and $by = yb$ for all $y \in G$.

   Thus,
   $$(ab)y = aby \overset{by=yb}{=} ayb \overset{ay=ya}{=} yab = y(ab)$$
   for all $y \in G$.

   So, $ab \in Z(G)$.

3. Let $c \in Z(G)$.
   Then, $cy = yc$ for all $y \in G$.
   So, $c^{-1}(cy)c^{-1} = c^{-1}(yc)c^{-1}$ for all $y \in G$.
   Thus, $yc^{-1} = c^{-1}y$ for all $y \in G$.
   Hence $c^{-1} \in Z(G)$.

By 1, 2, 3 we know that $Z(G) \trianglelefteq G$.

$\boxed{//}$